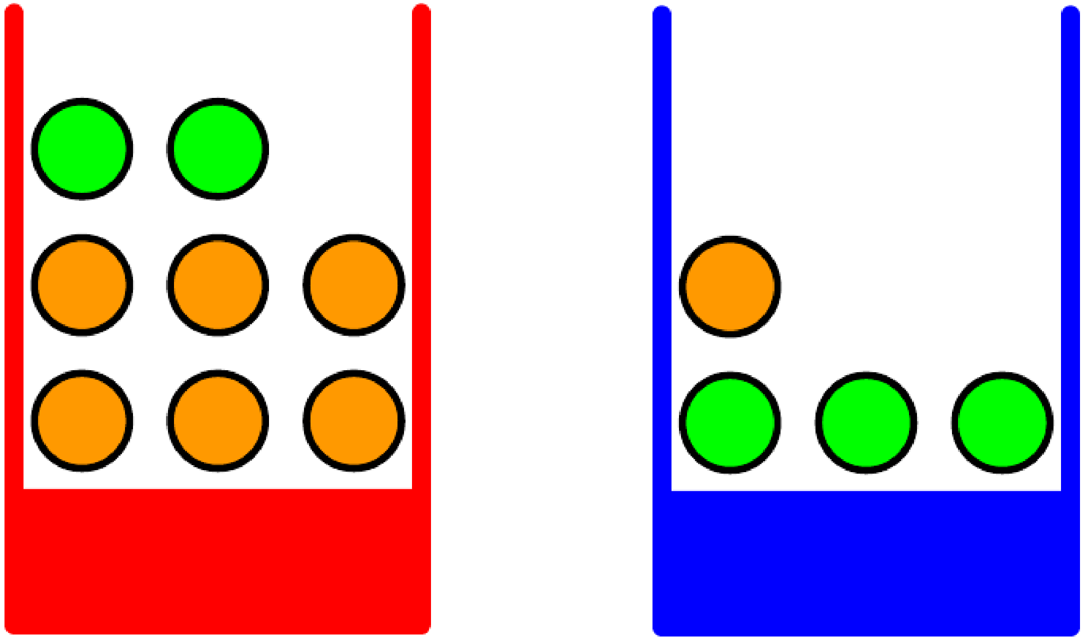
# Apples and oranges

– Randomly choose one box, and then randomly pick a piece of fruit from that box



$p(B = r) = 4/10$ $\quad$ $p(B = b) = 6/10$

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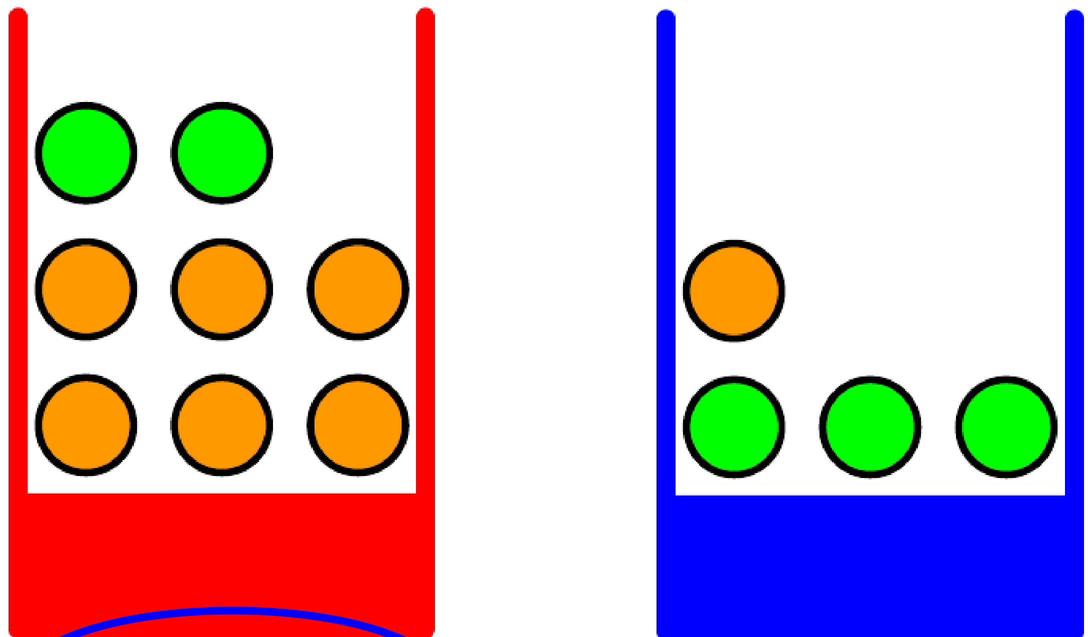

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probability we choose red Box

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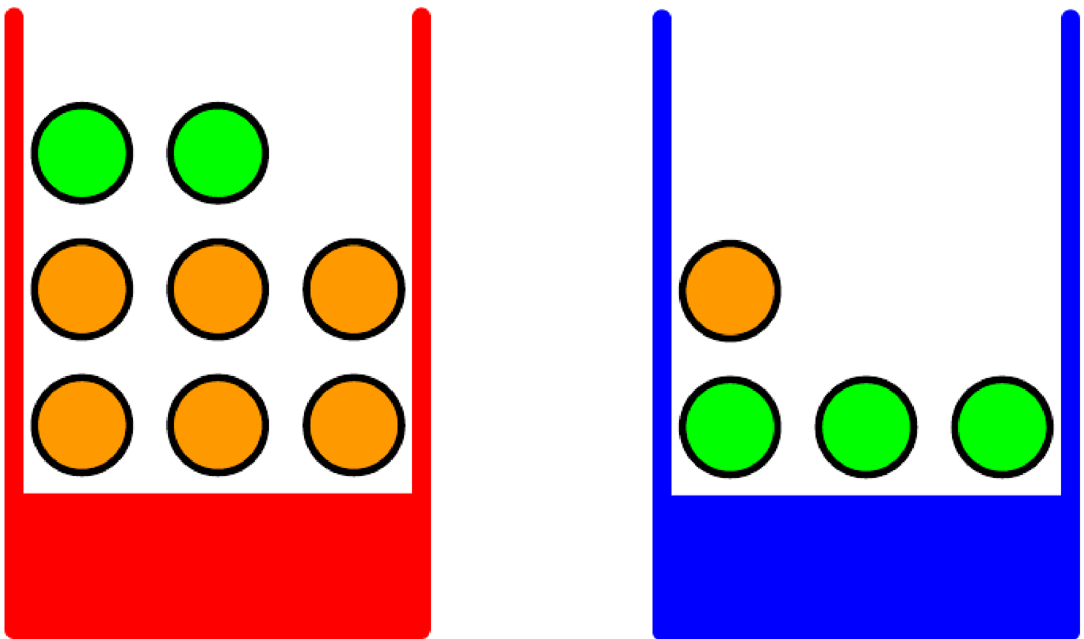

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probability we choose blue Box

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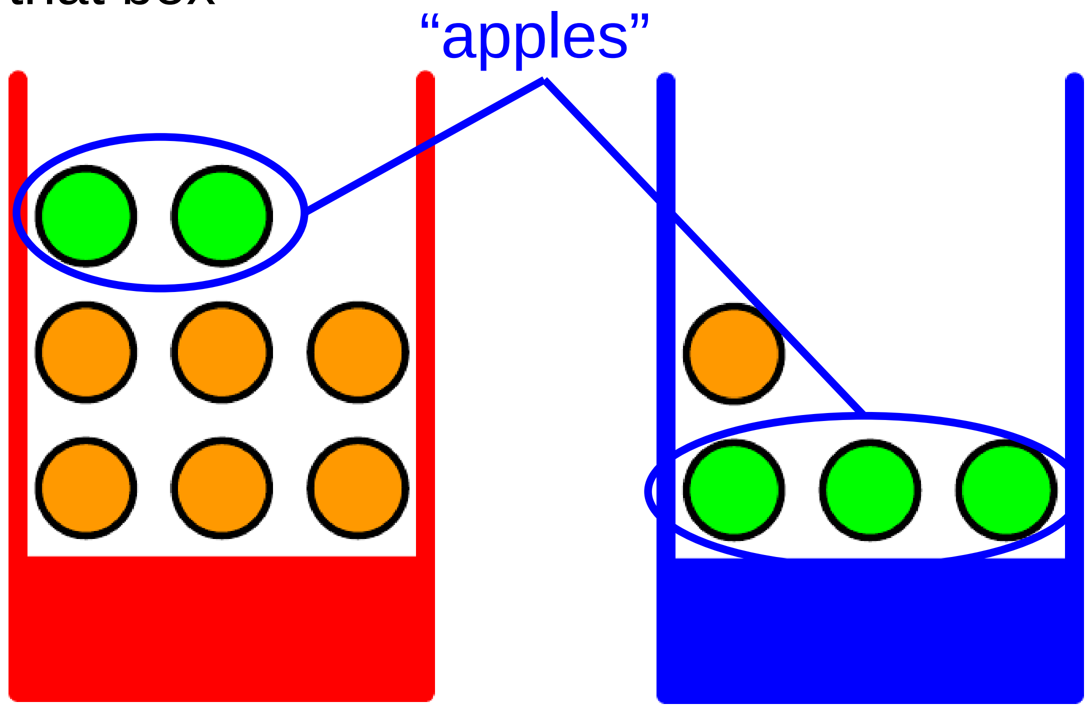


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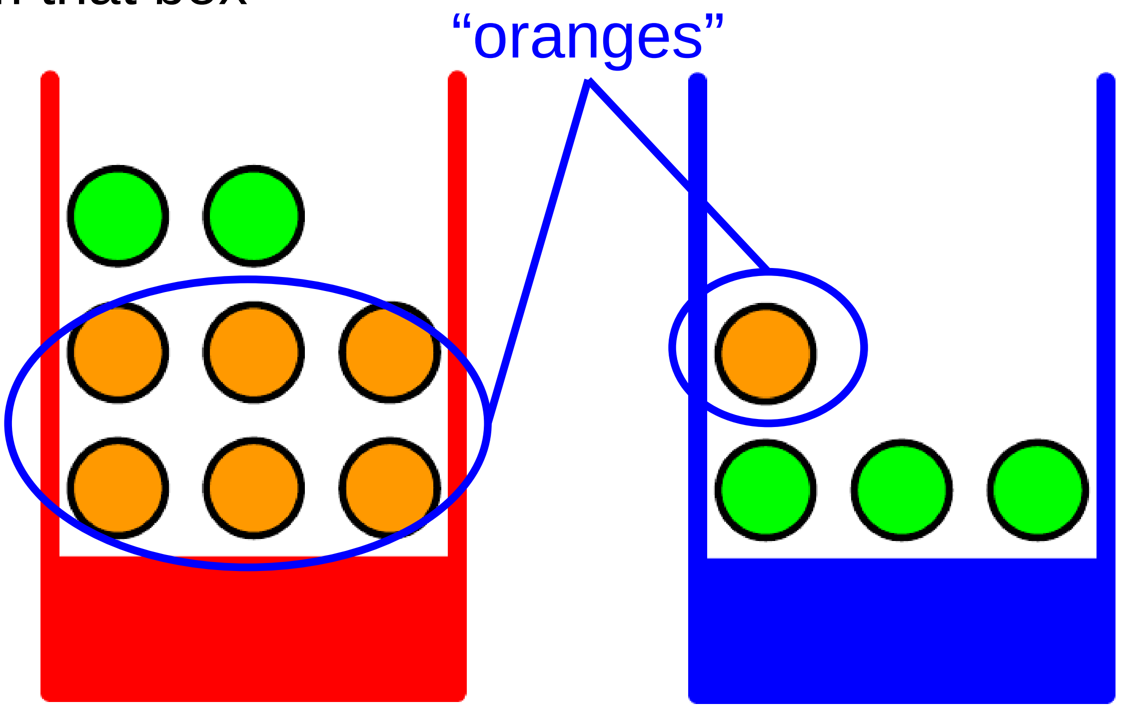


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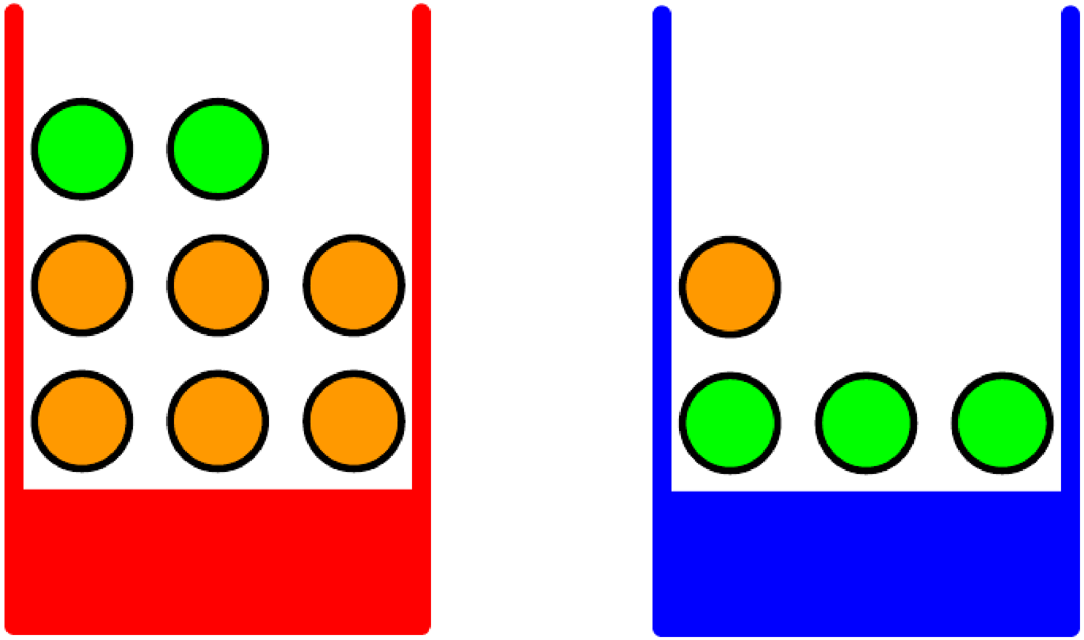


$p(B = r) = 4/10$ $p(B = b) = 6/10$

(1) What is the probability we get an apple?

(2) If the fruit we end up with is an orange, what is the probability we had chosen the red box?

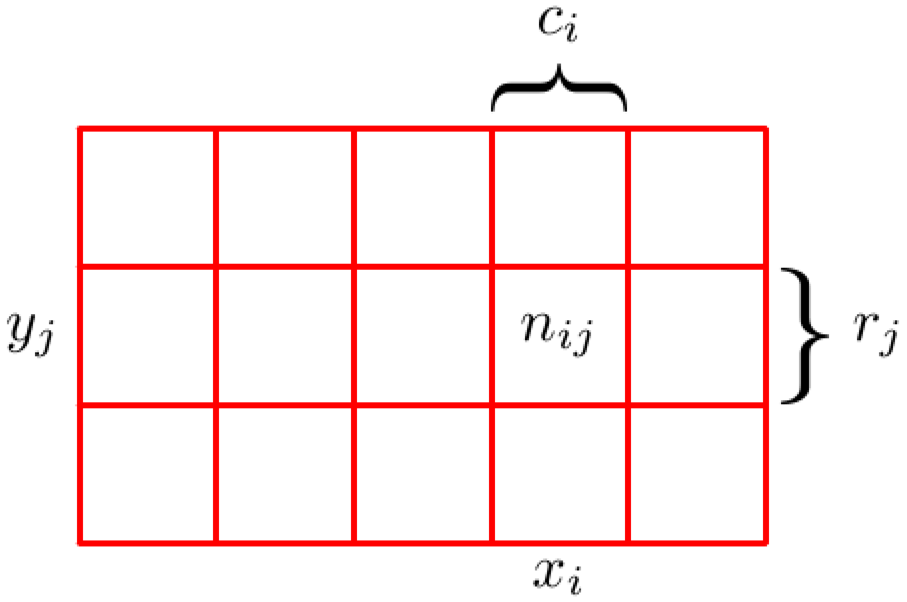
# The two rules of probability

- Two random variables:

  $X$ takes values $\{x_i\}$ where $i = 1, \ldots, M$

  $Y$ takes values $\{y_j\}$ where $j = 1, \ldots, L$

- Observe outcomes $(X, Y)$ of $N$ samples, with $N \to \infty$



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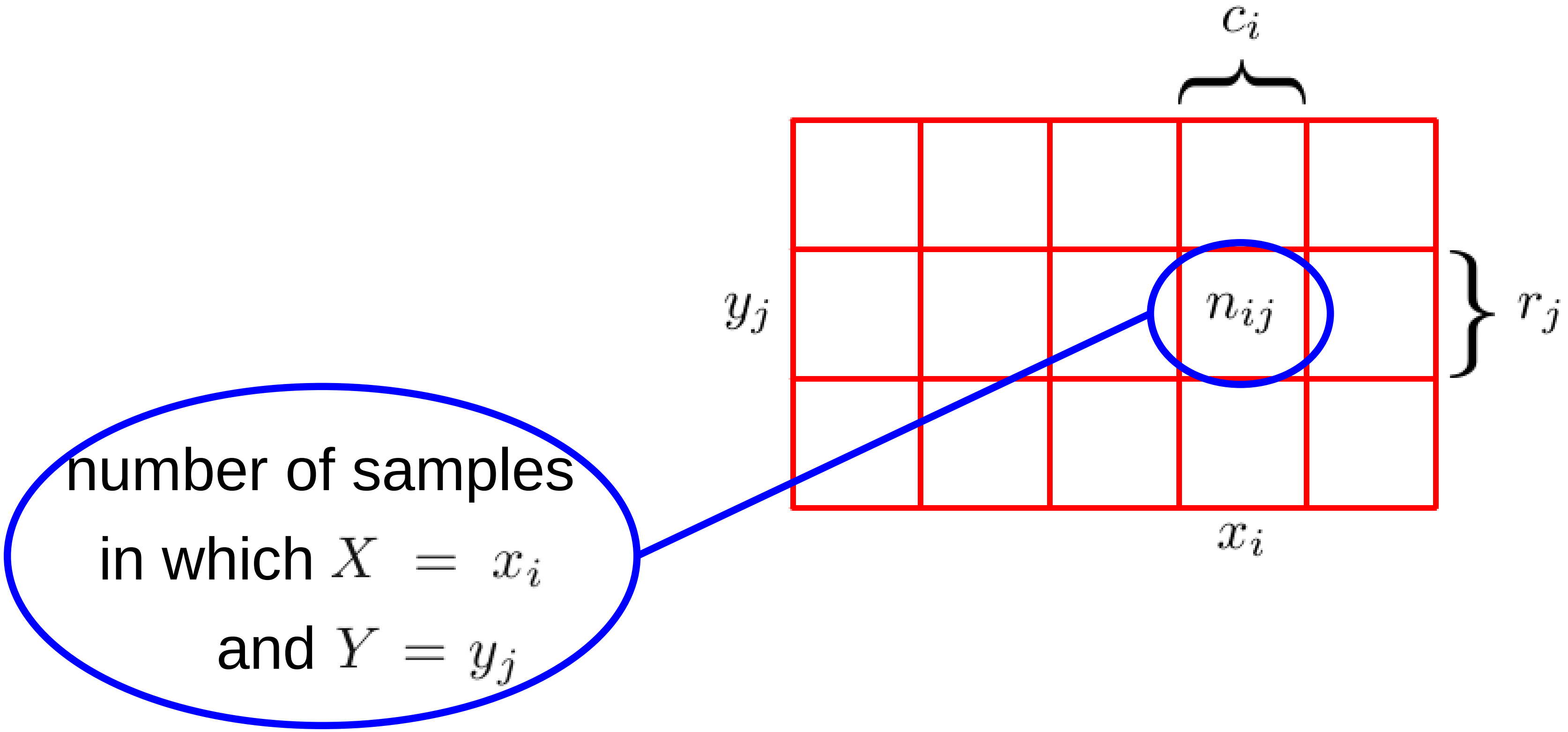

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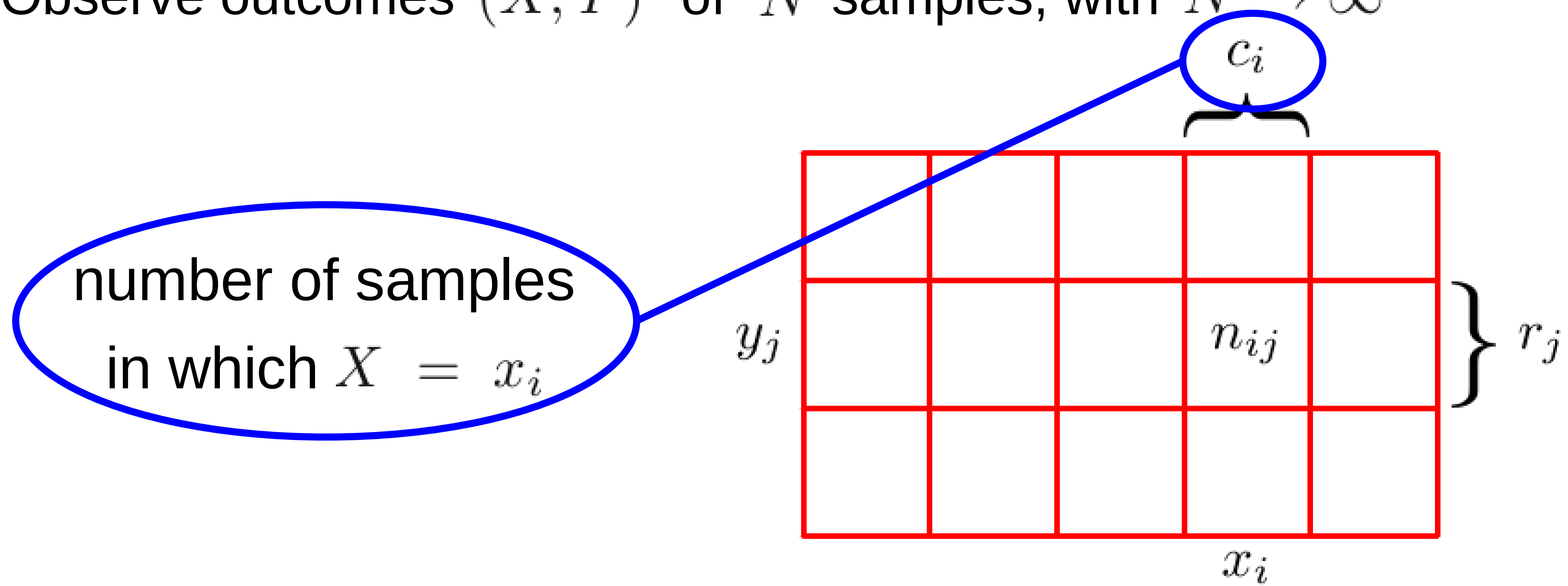

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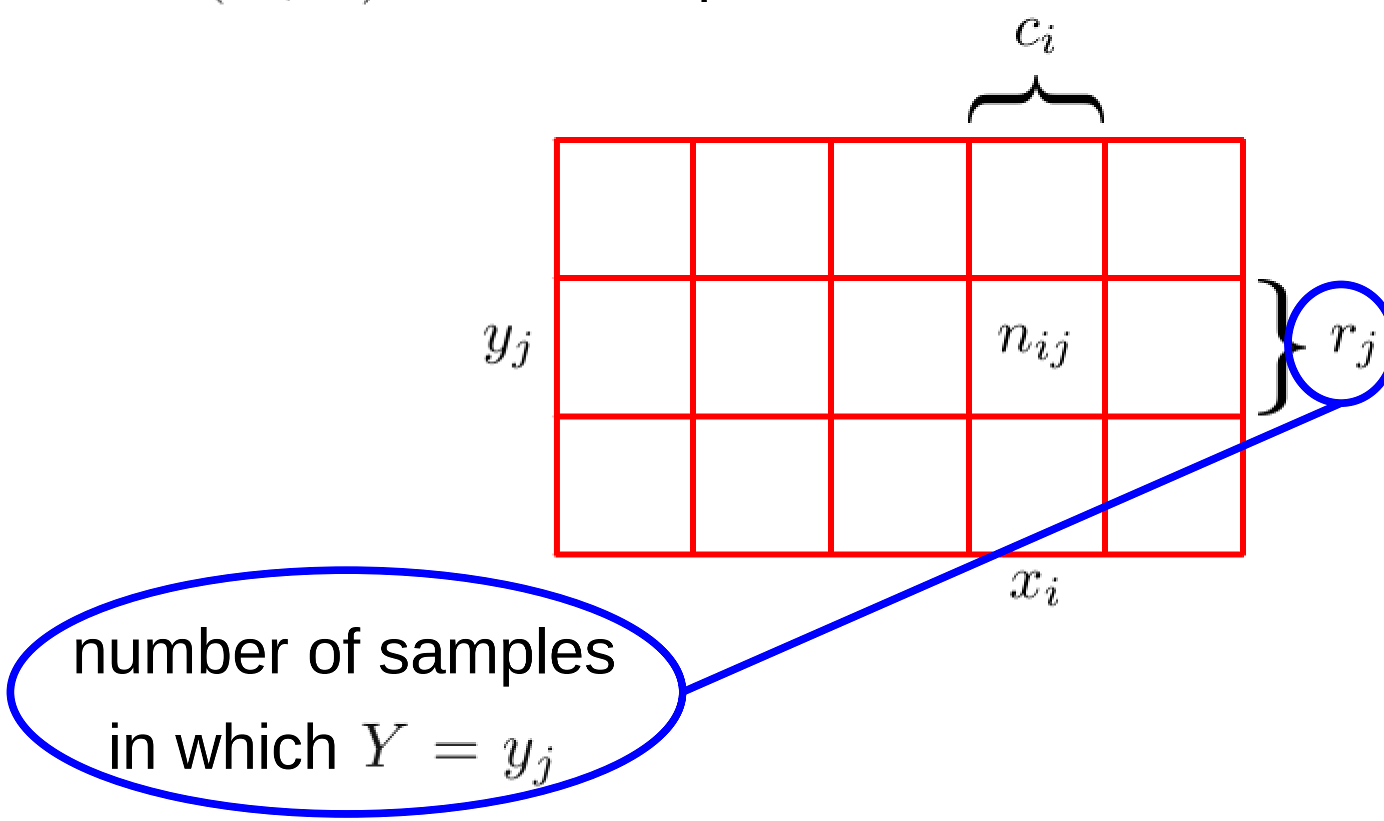

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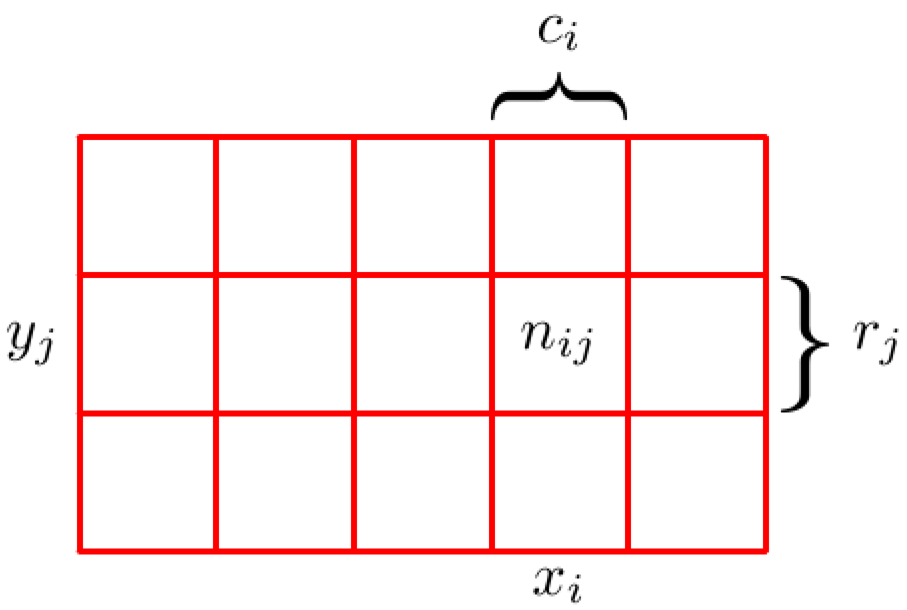
"joint probability"

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$



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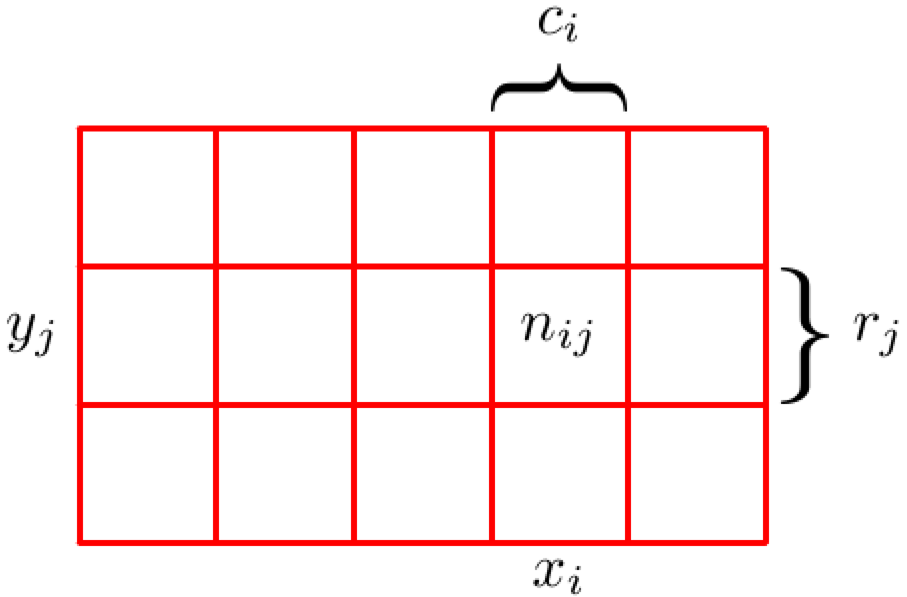


"marginal probability"

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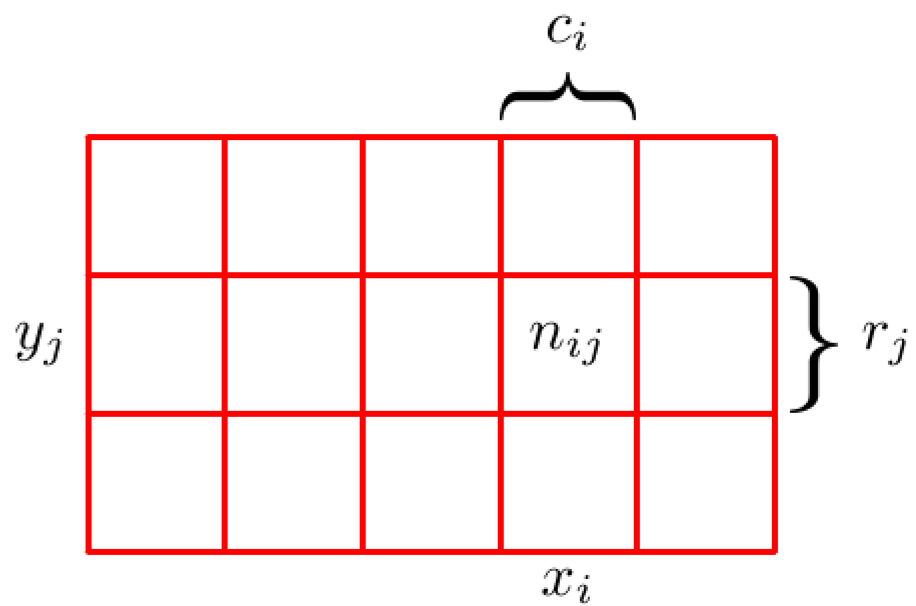


"marginal probability"

$$p(X = x_i) = \frac{c_i}{N} = \frac{\sum_j n_{ij}}{N} = \sum_{j=1}^{L} p(X = x_i, Y = y_j)$$

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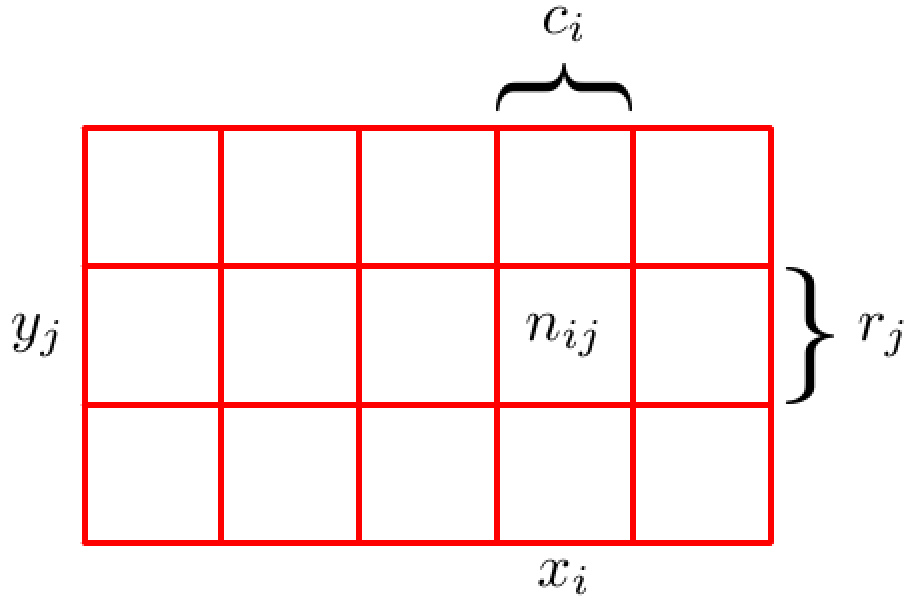


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$$p(X = x_i) = \frac{c_i}{N}$$

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"sum rule"

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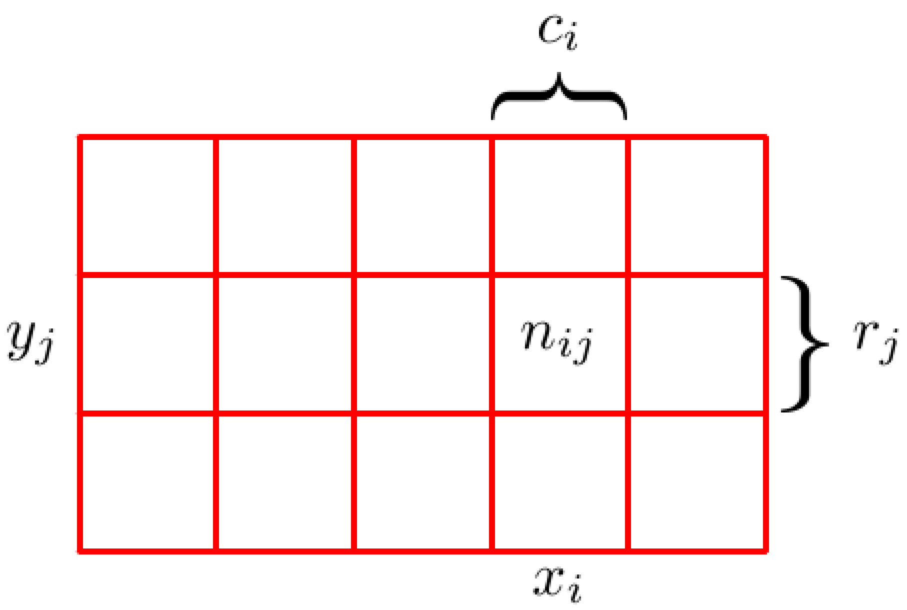
"conditional probability"

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$



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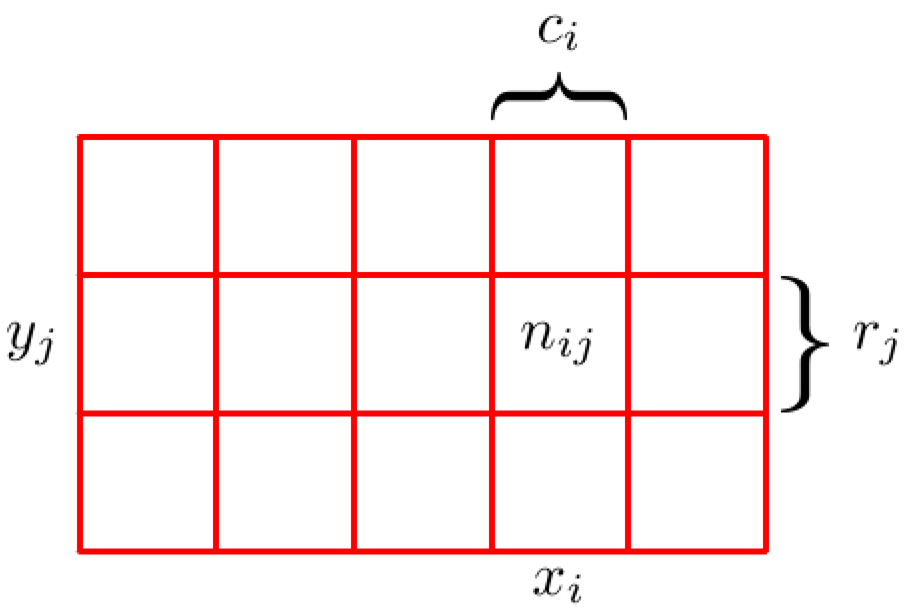

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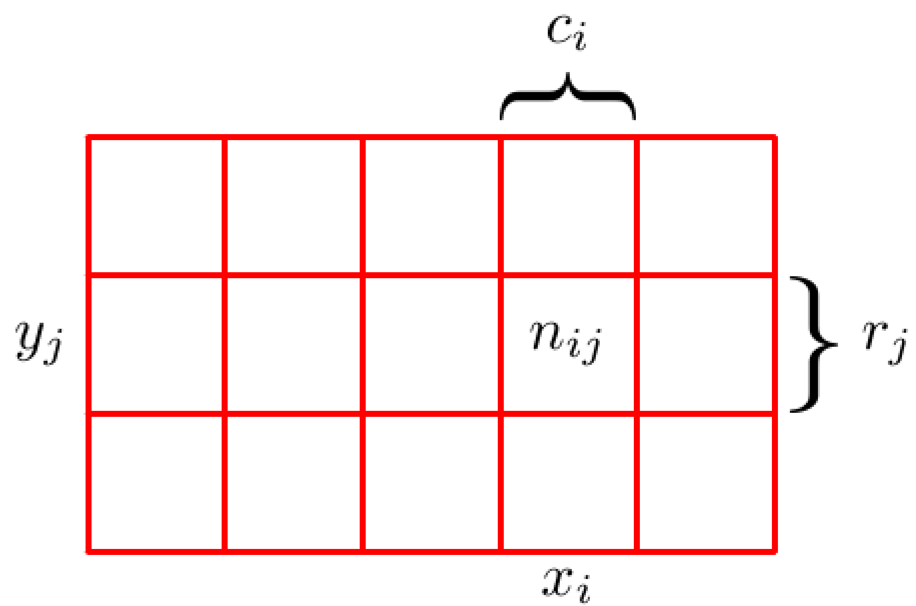
$$= \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N}$$

$$= \quad p(Y = y_j | X = x_i) p(X = x_i)$$



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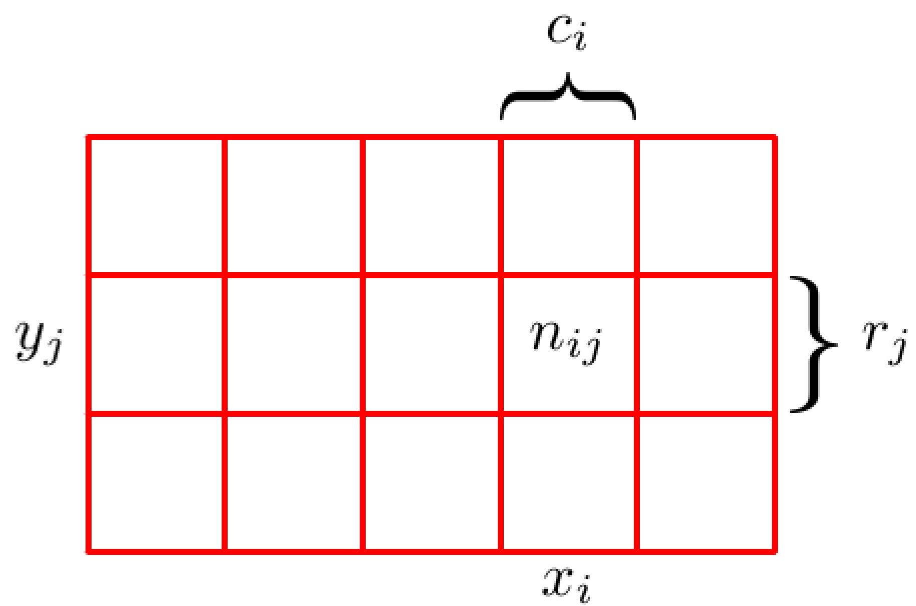



“joint probability”

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

“product rule”

$$= \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N}$$

$$= \quad p(Y = y_j | X = x_i) p(X = x_i)$$

# The two rules of probability

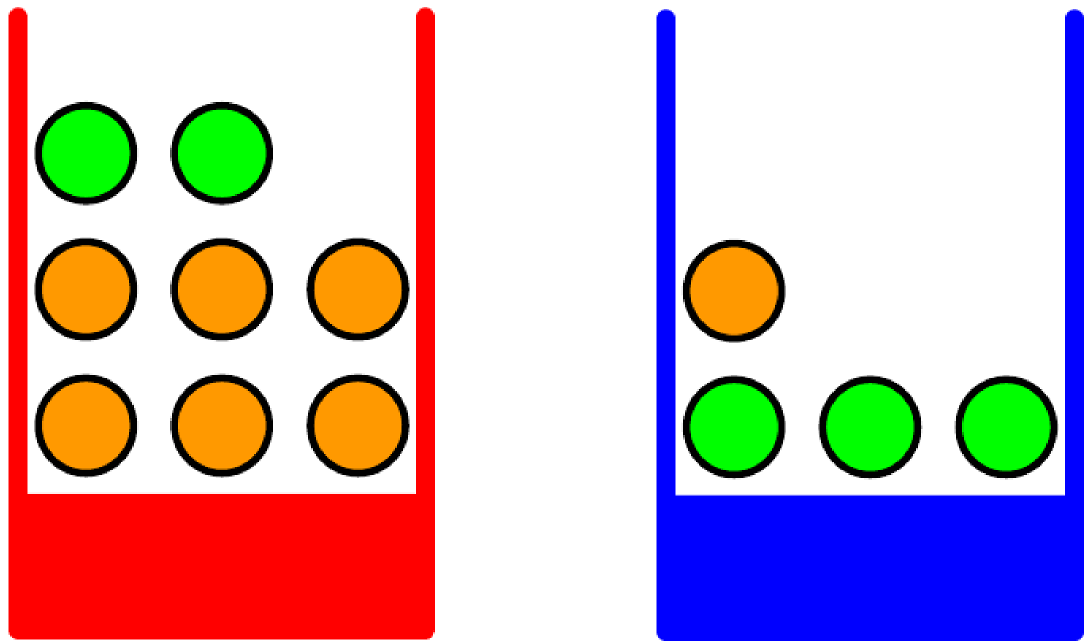
| | |
|---|---|
| **sum rule** | $p(X) = \sum_Y p(X, Y)$ |
| **product rule** | $p(X, Y) = p(Y\|X)p(X)$ |

– Direct result of product rule: "Bayes' rule"

$$p(Y|X) = \frac{p(X|Y)p(Y)}{p(X)}$$

# Apples and oranges

(1) What is the probability we get an apple?



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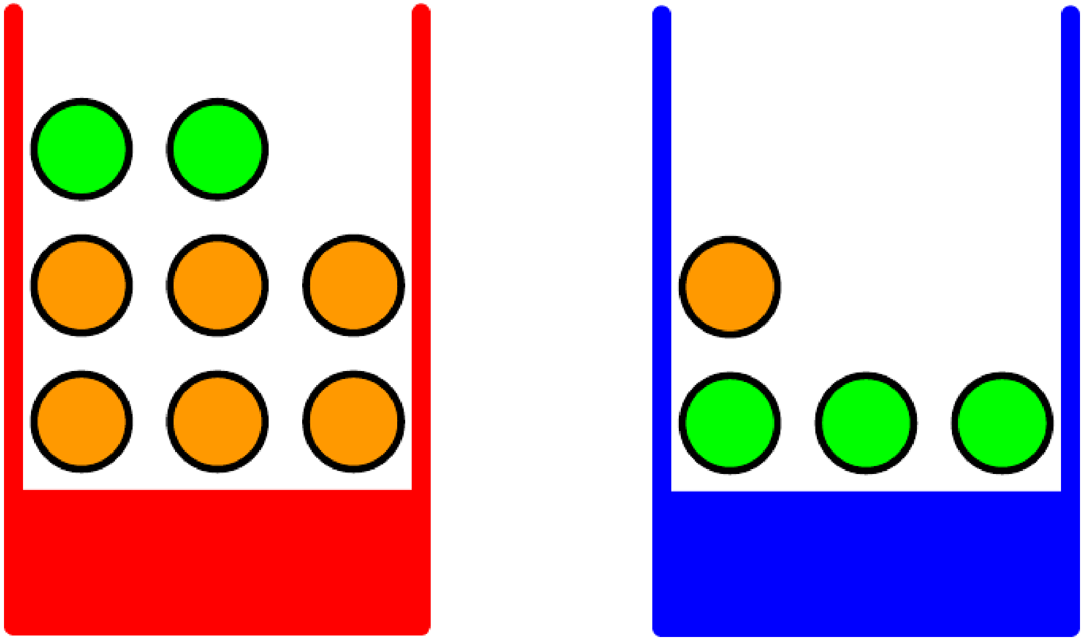

$p(B = r) = 4/10$

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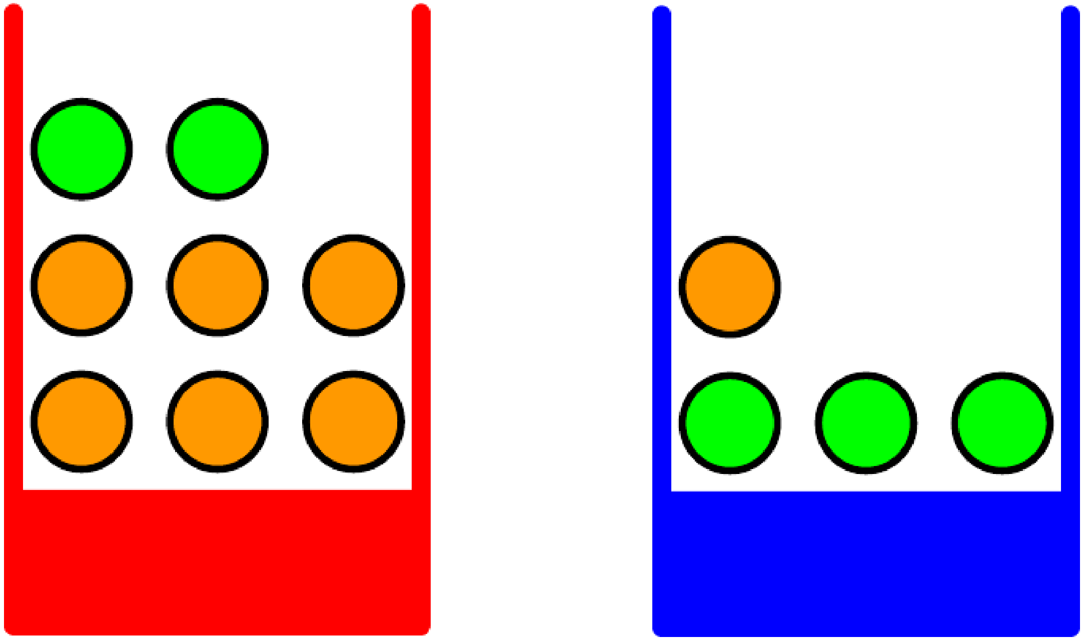
$$\begin{aligned} p(F = a) &= p(F = a|B = r)p(B = r) + p(F = a|B = b)p(B = b) \\ &= \frac{1}{4} \times \frac{4}{10} + \frac{3}{4} \times \frac{6}{10} = \frac{11}{20} \end{aligned}$$

# Apples and oranges

(2) If the fruit we end up with is an orange, what is the probability we had chosen the red box?



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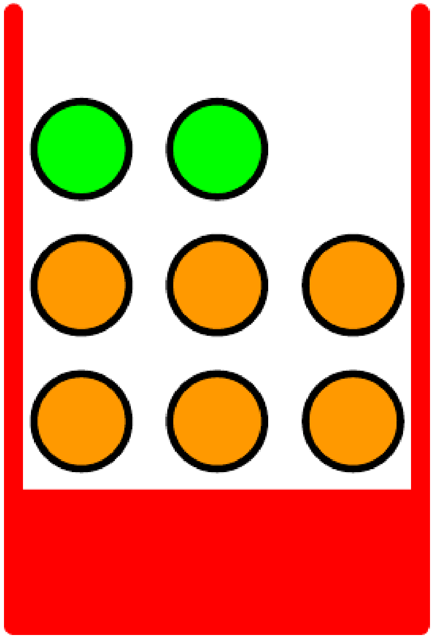
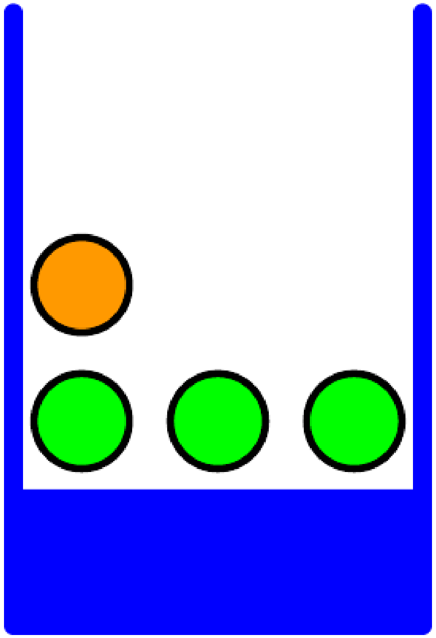


$p(B = r) = 4/10$

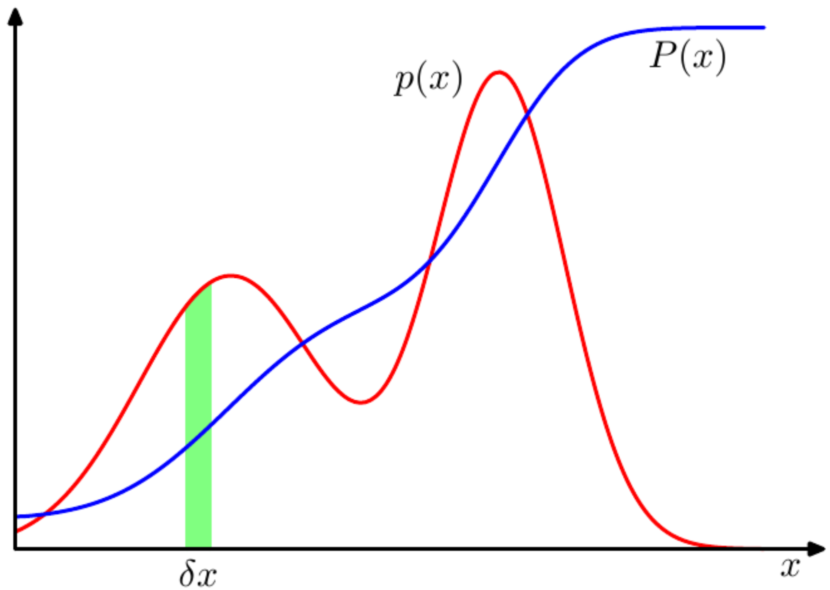


$p(B = b) = 6/10$

$$p(B = r|F = o) = \frac{p(F = o|B = r)p(B = r)}{p(F = o)}$$
$$= \frac{3}{4} \times \frac{4}{10} \times \frac{20}{9} = \frac{2}{3}$$

# Continuous variables

– $p(x)$ is called the *probability density* over $x$
if the probability that $x$ falls in the interval $(x, x+\delta x)$
is given by $p(x)\delta x$ for $\delta x \to 0$




$$p(x) \geqslant 0$$

$$\int_{-\infty}^{\infty} p(x)\,\mathrm{d}x = 1$$

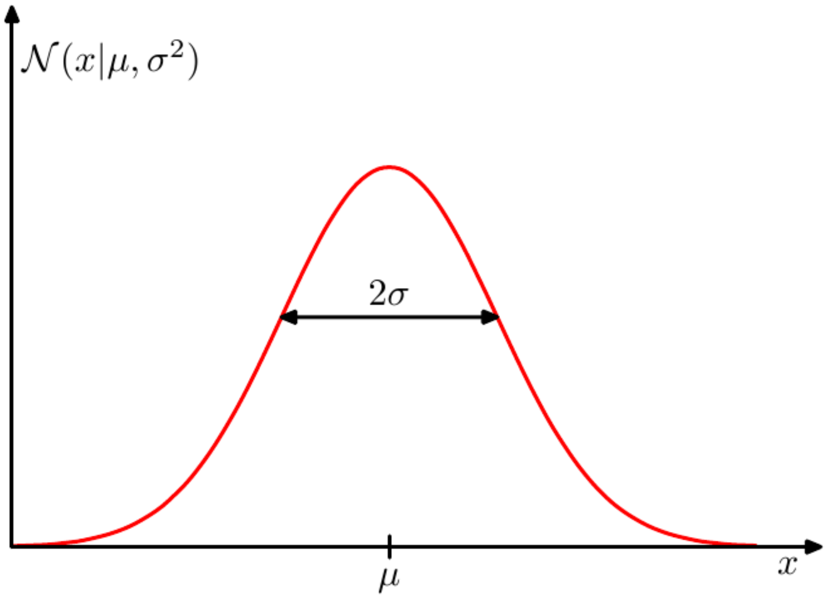
sum rule: $$p(x) = \int p(x,y)\,\mathrm{d}y$$

product rule: $$p(x,y) = p(y|x)p(x).$$

# Continuous variables

- Example: Gaussian distribution

$$\mathcal{N}\left(x|\mu,\sigma^2\right) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$




- Multivariate probability density: $p(\mathbf{x}) = p(x_1, \ldots, x_D)$

$$p(\mathbf{x}) \geqslant 0$$

$$\int p(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$$